Sign-changing solutions to overdetermined elliptic problems in bounded domains

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Outline

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Overdetermined elliptic problems

We consider semilinear elliptic problems in the form:

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \partial_\nu u = \text{constant} & \text{on } \partial\Omega. \end{cases} \tag{1}$$

These problems appear quite naturally in the study of free boundaries in many different phenomena in Physics, like in capillarity, elasticity and others.

Because of the two boundary conditions one does not expect, in general, generic existence results.

The first rigidity result is due to J. Serrin in 1971: if Ω is bounded and u is positive, then necessarily Ω is a ball and u is radially symmetric. The proof is based on the moving plane method.

The BCN Conjecture

The case of unbounded domains was first treated by Berestycki, Caffarelli and Nirenberg in 1997.

They show that Ω must be a half-plane under assumptions of asymptotic flatness of the domain.

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BCN Conjecture: If $u > 0$ is bounded and $\mathbb{R}^N \backslash \overline{\Omega}$ is connected, then Ω is either a ball B^N, a half-space, a generalized cylinder $B^k \times \mathbb{R}^{N-k}$, or the complement of one of them.

The BCN conjecture is false!

This conjecture was disproved for $N \geq 3$ by P. Sicbaldi: he builds solutions in domains obtained as a periodic perturbation of a cylinder (for $f(u) = \lambda u$, see [Sicbaldi '10]).

This construction works also for $N = 2$, but in this case $\mathbb{R}^2 \setminus \Omega$ is not connected.

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The same result is true for $f = 1$ [Fall, Minlend & Weth '17], and also for more general terms $f(u)$ [R., Sicbaldi & Wu '22].

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There are also nonradial solutions in exterior domains, even in dimension 2 ([Ros, R. & Sicbaldi '20]).

Overdetermined problems and CMC surfaces

A formal analogy with constant mean curvature surfaces has been observed:

- Serrin's result is the counterpart of Alexandrov's one on CMC hypersurfaces.
- Sicbaldi example has a natural analogue in the Delaunay CMC surface.

The case of epigraphs has also been studied [Farina & Valdinoci '10], [Del Pino, Pacard & Wei '15], [Wang & Wei '19], in connection with the Bernstein problem and the De Giorgi conjecture.

Overdetermined problems in other frameworks

1. **Overdetermined problems on manifolds**: [Pacard & Sicbaldi '09], [Delay& Sicbaldi '15], [Espinar & Mao '18], [Dominguez-Vazquez, Enciso & Peralta-Salas '19].

   Of special interest is the case of the sphere: [Kumaresan & Prajapat '98], [Fall, Minlend & Weth '18], [Espinar & Mazet '19], [R., Sicbaldi & Wu pp].
2. **Overdetermined problems on cones**: [Pacella & Tralli, '20], [Iacopetti, Pacella & Weth pp].

All previous results are concerned with positive solutions.

Outline

Sign-changing solutions to overdetermined problems

1. **Fluid equations**: Overdetermined problems appear also from stationary solutions of Euler equations, see for instance [Dominguez-Vazquez, Enciso & Peralta-Salas '21], [Hamel & Nadirashvili '21], [R. pp]. In this framework, the function u need not be positive, a priori.

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1. **Fluid equations**: Overdetermined problems appear also from stationary solutions of Euler equations, see for instance [Dominguez-Vazquez, Enciso & Peralta-Salas '21], [Hamel & Nadirashvili '21], [R. pp]. In this framework, the function u need not be positive, a priori.
2. **Schiffer Conjecture**: Let $\Omega \subset \mathbb{R}^N$ be a bounded regular domain, and $w : \Omega \to \mathbb{R}$ a non-constant solution to the problem:
$$\begin{cases} \Delta w + \lambda w = 0 & \text{in } \Omega, \\ w = c & \text{on } \partial\Omega, \\ \frac{\partial w}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$
Then Ω is a ball and w is radially symmetric.

   If we define $u = w - c$ we are led with a problem like (1) without any sign restriction on the function u.

A natural question

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The answer is no!

Theorem (R., preprint)

Let $N = 2$, 3 or 4. There exist bounded domains $\Omega \subset \mathbb{R}^N$ different from a ball such that the problem:

$$\begin{cases} -\Delta u = u - (u^+)^3 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \partial_\nu u = \text{constant} \neq 0 & \text{on } \partial\Omega, \end{cases}$$

admits a sign-changing solution.

Scheme of the proof

1. First we find a 1-parametric family of sign-changing radial solutions u_R of the problem:

$$\begin{cases} -\Delta u = u - (u^+)^3 & \text{in } B(R), \\ u = 0 & \text{on } \partial B(R). \end{cases}$$

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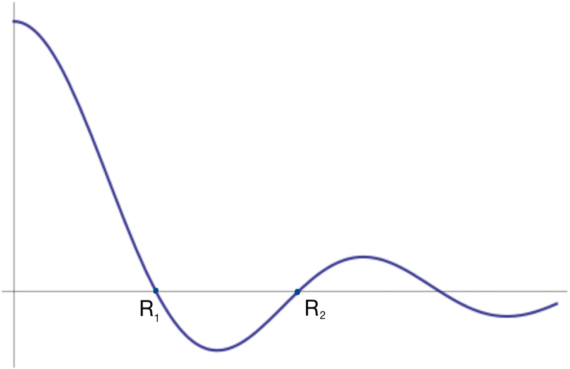
4. The nondegeneracy allows us to define a nonlinear operator $F(R, v)$ whose zeroes are solutions of our problem.
5. We prove a local bifurcation result for such operator F.

Outline

1. The radial solutions

The information about the radial eigenvalues of the laplacian is given by the radial solution of the problem $-\Delta U = U$.

The function U can be written by using Bessel functions of the first kind: $U(r) = r^{1-N/2} J_{N/2-1}(r)$, $r = |x|$.



1. The radial solutions

First we build the positive part of the solution. For any $p > 0$, we consider the Euler-Lagrange functional of the Allen-Cahn equation:

$$F : H_0^1(B(p)) \to \mathbb{R}, \; F(z) = \int_{B(p)} \frac{1}{2}|\nabla z|^2 + \frac{1}{4}(1 - z^2)^2$$

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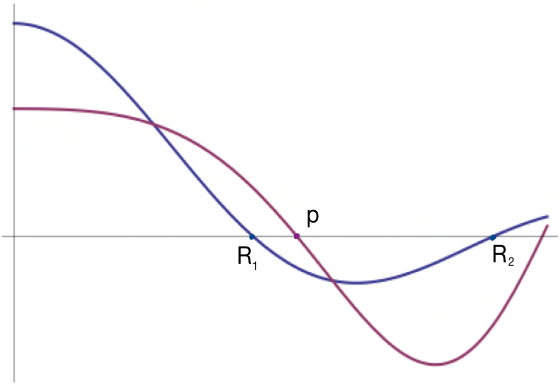
Clearly 0 is a solution, and its linearization is $-\Delta\phi - \phi$. But this operator is semipositive definite only if $p \leq R_1$.

For $p > R_1$ the minimizer is not trivial. We can assume that it is positive (and radial).

1. The radial solutions

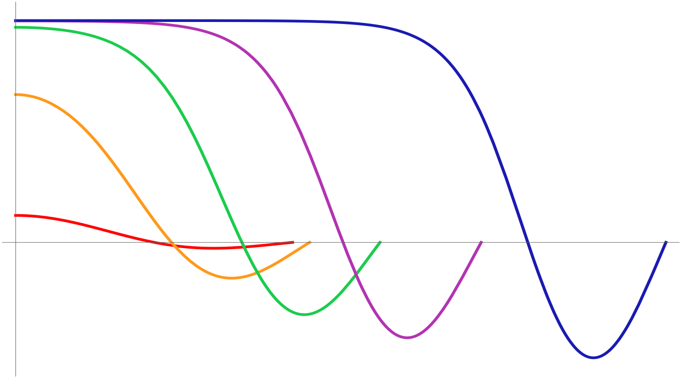
This solution continues with negative values, and hits again the x-axes at some $R > R_2$, by separation of zeroes of Sturm.



1. The radial solutions

For any $R > R_2$, there exists a sign-changing solution u_R, which vanishes at a point $p_R \in (0, R)$. Moreover, $\lim_{R \to R_2} u_R = 0$ and

$$\lim_{R \to +\infty} u_R(\cdot - p_R) = u_0, \quad u_0(r) = \begin{cases} -\tanh(\frac{r}{\sqrt{2}}) & r \leq 0, \\ -\frac{1}{\sqrt{2}} \sin(r) & r \in (0, \pi]. \end{cases}$$



2. Radial nondegeneracy

We define $L = -\Delta - 1 + 3(u_R^+)^2$ and $\bar{\lambda}_k$ its radial eigenvalues.

a) $\bar{\lambda}_1 < 0$. Indeed, take $\phi = u_R^-$. Then,

$$Q_D(\phi) := \int_{B(R)} L(\phi)\phi = 0.$$

If Q_D is semipositive definite, then u_R^- would be an eigenfunction, but $u_R^- = 0$ in $(0, p_R)$. Hence $\bar{\lambda}_1 < 0$.

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b) $\bar{\lambda}_2 > 0$. Take now $\bar{\phi}_1$, $\bar{\phi}_2$ eigenfunctions corresponding to $\bar{\lambda}_1$, $\bar{\lambda}_2$. Define $\phi = \alpha_1\bar{\phi}_1 + \alpha_2\bar{\phi}_2$ such that $\phi(p_R) = 0$. Then:

$$Q_D(\phi) = \int_{B(p_R)} L(\phi)\phi + \int_{B(R)\setminus B(p_R)} L(\phi)\phi > 0.$$

As a consequence $\bar{\lambda}_2 > 0$.

In particular u_R form a smooth 1-parametric family of solutions.

3. Nonradial degeneracies

Let us consider a symmetry group $G \subset O(N)$, and define λ_k the G-symmetric eigenvalues of L. Clearly $\lambda_1 = \bar{\lambda}_1 < 0$.

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If $\lambda_2 = \bar{\lambda}_2$, we are done. Otherwise observe that, as $R \to R_2$,

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$$Q_D(\phi) \sim \int_{B(R_2)} |\nabla \phi|^2 - \phi^2.$$

This operator becomes negative for some nonradial eigenfunctions. If we take G a group that excludes all those, then $\lambda_2 > 0$.

3. Nonradial degeneracies

b) For any group G, we can take R large enough so that $\lambda_2 < 0$.

Take as a test function $\phi = \xi(r - p_R)\vartheta(\theta)$, where

1. $\xi : (-\infty, \pi) \to \mathbb{R}$ has compact support, and
2. $\vartheta : \mathbb{S}^{N-1} \to \mathbb{R}$ is any G-symmetric spherical harmonic with eigenvalue γ.

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Clearly ϕ is orthogonal to $\bar{\phi}_1$. Moreover,

$$Q_D(\psi) = R^{N-1}\hat{Q}_D(\xi) + \gamma\, R^{N-3} \int \xi(r)^2\, dr + l.o.t.,$$

where $\hat{Q}_D : H_0^1(-\infty, \pi) \to \mathbb{R}$ is defined as:

$$\hat{Q}_D(\xi) = \int_{-\infty}^{\pi} |\xi'(r)|^2 - \xi^2 + 3(u_0^+)^2\xi^2.$$

3. Nonradial degeneracies

Recall that:

$$u_0(r) = \begin{cases} -\tanh(\frac{r}{\sqrt{2}}) & r \le 0, \\ -\frac{1}{\sqrt{2}}\sin(r) & r \in (0, \pi], \end{cases}$$

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Again, $\hat{Q}_D(u_0^-) = 0$ and $u_0^- = 0$ in $(-\infty, 0)$, hence $\hat{Q}_D$ achieves negative values. By density we can take a compactly supported function ξ with $\hat{Q}_D(\xi) < 0$.

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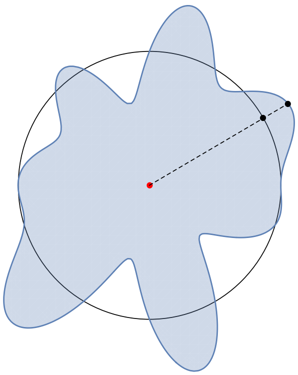
We define $\bar{R} > R_2$ as the first value for which $\lambda_2 = 0$: that is, $\lambda_2 > 0$ if $R \in (R_2, \bar{R})$ and $\lambda_2 = 0$ if $R = \bar{R}$.

From now on we restrict ourselves to $R \in (R_2, \bar{R})$.

4. The nonlinear Dirichlet-to-Neumann operator

Fix $R \in (R_2, \bar{R})$. Given a function $w : \mathbb{S}^{N-1} \longmapsto (0, \infty)$, let us denote $B(w)$ its radial graph,

$$B(w) := \left\{ x \in \mathbb{R}^N : \ |x| < w(x/|x|) \right\}.$$



4. The nonlinear Dirichlet-to-Neumann operator

By the Inverse Function Theorem, for all $v \in C_G^{2,\alpha}(\mathbb{S}^{N-1})$ small, there exists a positive solution $u = u(R, v)$ to the problem

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We define the Dirichlet-to-Neumann operator:

$$F(R, v) = \frac{\partial u}{\partial \nu} - \frac{1}{|\partial B(R+v)|} \int_{\partial B(R+v)} \frac{\partial u}{\partial \nu}\, dx,$$

Clearly, we are done if we prove the existence of nontrivial solutions of the equation $F(R, v) = 0$. From now on, we assume that v has 0 mean.

A necessary condition for bifurcation is that $D_v F(R, 0)$ becomes degenerate.

4. The nonlinear Dirichlet-to-Neumann operator

Proposition

$D_v F(R, 0) = c_R \, H_R(v)$, *where* c_R *is a constant and* H_R *is defined as:*

$$H_R(v) = \partial_\nu(\psi_v) + \frac{N-1}{R} \, v. \tag{2}$$

Here ψ_v *is a solution of the linear problem:*

$$\begin{cases} -\Delta\psi_v - \psi_v + 3(u_\rho^+)^2\psi_v = 0, & \text{in } B(R), \\ \psi_v = v & \text{on } \partial B(R). \end{cases}$$

Such solution exists and is unique by the Dirichlet nondegeneracy of the problem.

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Such solution exists and is unique by the Dirichlet nondegeneracy of the problem.

Moreover, if v has 0 mean then $\psi_v \in E$, where:

$$E = \{\phi \in H^1_G(B) : \int_B \phi(x) g(x)\, dx = 0 \ \ \forall\, g \in L^2_r(B)\}.$$

5. The local bifurcation

We study the quadratic form associated to H:

$$\int_{\partial B(R)} H(v)v = Q(\psi_v),$$

$$Q(\phi) = \int_{B(R)} \left(|\nabla \phi|^2 - \phi^2 + 3(u_\rho^+)^2 \phi^2 \right) + \frac{(N-1)}{R} \int_{\partial B} \phi^2.$$

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a) If $R \sim R_2$ and G is large enough, $Q|_E$ is positive definite.

$$Q(\phi) \sim \int_{B(R_2)} \left(|\nabla\phi|^2 - \phi^2 \right) \quad + \frac{N-1}{R_2} \int_{\partial B(R_2)} \phi^2$$
$$\geq \int_{B(R_2)} \left(|\nabla\phi|^2 - \phi^2 \right).$$

We are done if G excludes all nonradial eigenvalues smaller than 1 of the laplacian under Neumann boundary conditions in $B(R_2)$.

5. The local bifurcation

b) If $R \sim \bar{R}$, then $Q|_E$ becomes negative.
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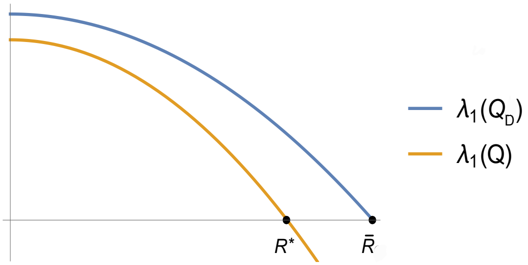
Observe that $Q_D|_E$ is nothing but $Q|_E$ restricted to functions which vanish at $\partial B(R)$.

Recall that $Q_D|_E$ is positive definite for $R \in (R_2, \bar{R})$ and Q_D is positive semidefinite for $R = \bar{R}$.



Then the operator H becomes degenerate at some $R^* \in (R_2, \bar{R})$!

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The kernel of H is formed by functions in the form:

$$\phi = f(r)\vartheta(\theta),$$

where we are using spherical coordinates and:

1. $f(r)$ is the solution of an ODE problem, which is unique;
2. ϑ is a nontrivial eigenfunction of $-\Delta$ on $\mathbb{S}^{N-1}$ at the first G-symmetric eigenvalue σ.

If such eigenvalue σ has odd multiplicity, then the Krasnoselskii bifurcation theorem can be applied and we obtain local bifurcation.

On the symmetry group G

In sum, we need a symmetry group $G \subset O(N)$ such that:

1. The G-symmetric nonradial Dirichlet eigenvalues of Δ in $B(R_2)$ are all bigger than 1.
2. The G-symmetric nonradial Neumann eigenvalues of Δ in $B(R_2)$ are all bigger than 1.
3. The first G-symmetric eigenvalue σ of Δ on $\mathbb{S}^{N-1}$ has odd multiplicity.

If $N = 2,\ 3,\ 4$ or 5, this is satisfied if $\sigma = k(k + N - 2)$ with $k \geq 5$ and with odd multiplicity.

For $N = 6$ we need $k \geq 6$.

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3. In $\mathbb{R}^4$ there exists a regular polytope called hyper-icosahedron, with 600 tetrahedral cells and 120 vertices.

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   Its group of rotations satisfies that $k = 12$ and its multiplicity is 1 ([Nelson & Widom '84]).

For $N \geq 5$ the only regular polytopes are the hyper-tetrahedron, the hyper-cube and the hyper-octahedron, and their symmetry groups are too small.

Thank you for your attention!